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Invertible Matrices Over Distributive Pseudo-lattices*

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Abstract: A complete description of the invertible matrices over a distributive pseudo-lattice is given, some necessary and sufficient conditions for a matrix over a distributive pseudo-lattices to be invertible are obtained. Moreover, it is proved that a matrix is invertible if and only if it is a permutation matrix over an integral distributive pseudo-lattice. These results can be regarded as generalizations of the previous results on the invertible matrices over distributive lattices and commutative inclines.

Keywords: distributive pseudo-lattices; invertible matrix; orthogonal

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1 Introduction

Invertible matrices include some important kinds of matrices. Luce[1] showed that a matrix over a Boolean algebra of at least two elements is invertible if and only if it is an orthogonal matrix. Zhao[2] proved that a fuzzy square matrix is invertible if and only if it is a permutation matrix. Give'on[3] developed the theory of invertible lattice matrices, which generalized the result of Luce[1]. Zhao[4-5] discussed the conditions to invertibility of matrices over a kind of Brouwerian lattices and an arbitrary distributive lattice, respectively. Recently, Han et al[6] gave a complete description of the invertible incline matrices, they investigated some necessary and sufficient conditions for an incline matrix to be invertible over inclines.

In the present work, we consider the invertible matrices over general distributive pseudo-lattices. We give a complete description for the invertible matrices and obtain some necessary and sufficient conditions for a matrix over a distributive pseudo-lattice to be invertible. Also it is proved that a matrix is invertible if and only if it is a permutation matrix over Z, where Z is an integral distributive pseudo-lattice. These results can be regarded as the generalizations of the previous results on the invertible matrices over distributive lattices and commutative inclines[1-6].

2 Definitions and notations

In this section, we give some necessary definitions and notations. And we also give an example for distributive pseudo-lattice. In the following, for the definitions and notation of invertible matrix and permutation matrix etc, readers can refer to [6].

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Definition 2.1 A nonempty set L with additive "+" and multiplicative "$\cdot$" is called a distributive pseudo-lattice if the following conditions hold:

1) $(L, +)$ is a semi-lattice;
2) $(L, \cdot)$ is a non-commutative semi-group;
3) $x(y+z) = xy + xz,\ (y+z)x = yx + zx$ for all $x,\ y,\ z \in L$;
4) $x + xy = x + yx = x$ for every $x,\ y \in L$.

In a distributive pseudo-lattice L, we define a relation "$\leq$" as follows: for every $x,\ y \in L,\ x \leq y \Longleftrightarrow x + y = y$. Evidently, the relation "$\leq$" is a partially ordering over L. Besides, for every $x,\ y \in L$, we have $xy \leq x$ and $yx \leq x$. We assume that L has the additive identity 0 and the multiplicative identity 1. That is, for every $x \in L$, we have $0x = x0 = 0,\ 1x = x1 = 1$. Clearly, 1 is the greatest element (the unit) of L, and 0 is the least element (the zero) of L.

Clearly, Boolean algebras, Fuzzy algebras, distributive lattices and commutative inclines are distributive pseudo-lattices.

Example 2.1 Let Ω be the set $\{0, a, b, c\}$. Operators addition "$\oplus$" and multiplication "$\otimes$" are defined over Ω as the following Tables 1 and 2.

Table 1: addition "$\oplus$"

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

Table 2: multiplication "$\otimes$"

$\otimes$	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

It is easy to check that $(\Omega,\ \oplus,\ \otimes)$ is a distributive pseudo-lattice.

Definition 2.2 Let $A \in M_{m\times n}(L)$ $(n \geq m)$. The permanent of A is defined by

$$per(A) = \sum_{\sigma\in\phi(\underline{m},\underline{n})} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{m\sigma(m)},$$

where $\underline{m}$ and $\underline{n}$ denote the sets $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$, respectively, $\phi(\underline{m},\ \underline{n})$ is a set of all injection mappings from the set $\underline{m}$ to the set $\underline{n}$.

Definition 2.3 If $ab = 0(ba = 0)$, then a and b is said to be left orthogonal (right orthogonal). If a and b is both left and right orthogonal, then a and b is said to be orthogonal. If $a_{ik}a_{il} = a_{il}a_{ik} = 0$ $(i \in \underline{m},\ k,\ l \in \underline{n},\ k \neq l)$, then A is said to be rows orthogonal; if $a_{kj}a_{lj} = a_{lj}a_{kj} = 0$ $(j \in \underline{n},\ k,\ l \in \underline{m},\ k \neq l)$, then A is said to be columns orthogonal.

Definition 2.4 If $a + b = 1$, then b and a are said to be mutual complement. If a and b are both orthogonal and mutual complement, then a and b are said to be mutual orthogonal complement, and $b(a)$ is the orthogonal complement of $a(b)$.

Obviously, if b is the orthogonal complement of a, then a is also the orthogonal complement of b, i.e., a and b are mutual orthogonal complement.

If

$$\sum_{k=1}^{n} a_{ik} = 1,$$

for every $i \in \underline{m}$, then the rows of A are said to mutually complement, if

$$\sum_{k=1}^{m} a_{kj} = 1,$$

for every $j \in \underline{n}$, then the columns of A are said to mutually complement. If the rows (columns) of A are both orthogonal and mutually complement, then the rows (columns) of A are said to mutual orthogonal complement.

Definition 2.5 A distributive pseudo-lattice L is integral, if there do not exist nonzero elements a, $b \in L$ such that a and b are mutually orthogonal complement. Denote Z a integral distributive pseudo-lattice L.

3 Invertible matrices over distributive pseudo-lattices

In this section, we give a description for invertible matrices over a distributive pseudo-lattice and obtain some necessary and sufficient conditions for the matrices to be invertible. Also it is proved that a matrix is invertible if and only if it is a permutation matrix over Z.

Proposition 3.1 If $A \in M_n(L)$ is invertible, then $AA^T = A^TA = I_n$.

Proof The fact that matrix A is invertible implies that there exists $X \in M_n(L)$ such that $XA = AX = I_n$. Let $A = (a_{ij})$, $X = (x_{ij})$. Then $AX = I_n$ implies that when $i \neq j$,

$$\sum_{k=1}^{n} a_{ik}x_{kj} = 0,$$

i.e., $a_{ik}x_{kj} = 0$ $(i \neq j, k \in \underline{n})$, and when $i = j$,

$$\sum_{k=1}^{n} a_{ik}x_{ki} = 1.$$

Similarly, $XA = I$ implies that $x_{ik}a_{kj} = 0$ $(i \neq j, k \in \underline{n})$, when $i = j$,

$$\sum_{k=1}^{n} x_{ik}a_{ki} = 1,$$

Hence, for every $i \in \underline{n}$, we have

$$\sum_{k=1}^{n} a_{kj} \geq \sum_{k=1}^{n} a_{ik}x_{ki} = 1, \quad \sum_{k=1}^{n} x_{ik} \geq \sum_{k=1}^{n} x_{ik}a_{ki} = 1,$$

that is

$$\sum_{k=1}^{n} a_{ki} = 1, \quad \sum_{k=1}^{n} x_{ik} = 1.$$

Therefore,

$$\begin{aligned} a_{ji} &= a_{ji}\cdot 1 = a_{ji}\sum_{k=1}^{n} x_{ik} = \sum_{k=1}^{n} a_{ji}x_{ik} = a_{ji}x_{ij} + \sum_{k\neq j} a_{ji}x_{ik} = a_{ji}x_{ij} + 0 \\ &= a_{ji}x_{ij} + \sum_{k\neq j} a_{ki}x_{ij} = \sum_{k=1}^{n} a_{ki}x_{ij} = \Big(\sum_{k=1}^{n} a_{ki}\Big)x_{ij} = 1\cdot x_{ij} = x_{ij}. \end{aligned}$$

Consequently, $X = A^T$, that is, $AA^T = A^TA = I_n$.

Obviously, when A is invertible, A^T is also invertible.

Theorem 3.1 The matrix $A \in M_n(L)$ is invertible if and only if the rows and columns of A are mutually orthogonal complement.

Proof "$\Rightarrow$": If $A \in M_n(L)$ is invertible, by Proposition 3.1, $AA^T = A^TA = I_n$, which implies

$$\sum_{k=1}^{n} a_{ik} \geq \sum_{k=1}^{n} a_{ik}^2 = 1, \quad \sum_{k=1}^{n} a_{kj} \geq \sum_{k=1}^{n} a_{kj}^2 = 1,$$

that is

$$\sum_{k=1}^{n} a_{ik} = 1, \quad \sum_{k=1}^{n} a_{kj} = 1,$$

and when $k \neq l$ for $i,\ j,\ k,\ l \in \underline{n}$,

$$a_{ik}a_{il} = 0, \quad a_{kj}a_{lj} = 0.$$

Consequently, the the rows and columns of A are mutually orthogonal complement.

"$\Leftarrow$": If

$$\sum_{k=1}^{n} a_{ik} = 1, \quad \sum_{k=1}^{n} a_{kj} = 1,$$

and $a_{ik}a_{jk} = 0$ for $i \neq j,\ i,\ j,\ k \in \underline{n}$, then

$$a_{ij}\sum_{k=1}^{n} a_{kj} = a_{ij}, \quad a_{ij}^2 = a_{ij}.$$

Hence

$$\sum_{k=1}^{n} a_{ik}^2 = \sum_{k=1}^{n} a_{ik} = 1,$$

Therefore,

$$AA^T = \Big(\sum_{k=1}^{n} a_{ik}a_{jk}\Big)_n = I_n.$$

If

$$\sum_{k=1}^{n} a_{ik} = 1, \quad \sum_{k=1}^{n} a_{kj} = 1,$$

and when $i \neq j, a_{ki}a_{kj} = 0$.

In the similar way, we can prove that $A^TA = I_n$.

Consequently, A is invertible. This completes the proof.

Theorem 3.2 $A \in M_n(L)$ is invertible if and only if $A^{[n]} = I_n$, where $[n]$ denotes the smallest common multiple of the integers $1, 2, \cdots, n$.

Proof "$\Rightarrow$": If $A \in M_n(L)$ is invertible, then by Proposition 3.1 and Theorem 3.1, we have $AA^T = A^TA = I_n$ and $a_{ik}a_{il} = 0$, $a_{kj}a_{lj} = 0$ for $i, j, k, l \in \underline{n}$, $k \neq l$. Now let $A^{[n]} = (a_{ij}^{[n]})$. In the following, we consider the multiplicative product

$$a_{ij}^{[n]} = a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_{k+1}}a_{i_{k+1}i_{k+2}}\cdots a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j}.$$

When $n = 2$, the identity $1 = (a_{11} + a_{12})(a_{21} + a_{22}) = a_{11}a_{22} + a_{12}a_{21}$ implies $a_{11}a_{22} + a_{12}a_{21} = 1$.

Since

$$\begin{aligned} a_{21}a_{11} + a_{22}a_{21} &= (a_{21}a_{11} + a_{22}a_{21}) \cdot 1 = (a_{21}a_{11} + a_{22}a_{21})(a_{11}a_{22} + a_{12}a_{21}) \\ &= a_{21}a_{11}a_{11}a_{22} + a_{22}a_{21}a_{11}a_{22} + a_{21}a_{11}a_{12}a_{21} + a_{22}a_{21}a_{12}a_{21} = 0, \end{aligned}$$

we have $a_{21}a_{11} + a_{22}a_{21} = 0$. Similarly, we can prove that $a_{11}a_{12} + a_{12}a_{22} = 0$.

Hence

$$\begin{aligned} A^{[n]} &= A^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & 0 \\ 0 & a_{21}a_{12} + a_{22}^2 \end{pmatrix} \leq I_n. \end{aligned}$$

When $n > 2$, it follows from $[n] - 1 > n$ that j in $a_{ij}^{[n]}$ coincides with one of $i_1, i_2, \cdots, i_{[n]-1}$. If j coincides with the $([n]-1)$-th index of $i_1, \cdots, i_{[n]-1}$, i.e., $j = i_{[n]-1}$, then

$$\begin{aligned} &a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} \neq 0, \quad \text{iff} \quad i_{[n]-1} = i_{[n]-2}\cdots, \\ &a_{ij}^{[n]} = a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_{k+1}}a_{i_{k+1}i_{k+2}}\cdots a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} \neq 0 \end{aligned}$$

iff

$$i = i_1 = i_2 = \cdots = i_{[n]-1} = i_{[n]-2} = j.$$

If $j \neq i_{[n]-1}$ and $j = i_{[n]-2}$, then $a_{i_{[n]-3}i_{[n]-2}}a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} \neq 0$, iff $i_{[n]-3} = i_{[n]-1}$. Since $[n]$ is multiple of 2, then $a_{ij}^{[n]} = a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_{k+1}}a_{i_{k+1}i_{k+2}}\cdots a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} \neq 0$, iff $i_{[n]-1} = i_{[n]-3} = i_{[n]-5} = \cdots = i_{[n]-(2k-1)} = \cdots = i_1, j = i_{[n]-2} = i_{[n]-4} = \cdots = i_{[n]-(2k)} = \cdots = i$.

If $j \neq i_{[n]-1}, i_{[n]-2}, \cdots, i_{[n]-n-1}$ and $j = i_{[n]-n}$, then in a similar way, we can prove that

$$a_{ij}^{[n]} = a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_{k+1}}a_{i_{k+1}i_{k+2}}\cdots a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} \neq 0,$$

iff the following conditions hold.

$$\begin{aligned}
&i = i_n = i_{2n} = \cdots = j,\\
&i_1 = i_{n-1} = i_{2n-1} = \cdots = i_{[n]-1},\\
&\cdots\cdots\\
&i_k = i_{n-k} = i_{2n-k} = \cdots = i_{[n]-k},\\
&\cdots\cdots
\end{aligned}$$

Consequently, we have $a_{ij}^{[n]} = a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_{k+1}}a_{i_{k+1}i_{k+2}}\cdots a_{i_{[n]-2}i_{[n]-1}}a_{i_{[n]-1}j} = 0$ with $i \neq j$, for all $i,\ j \in \underline{n}$. Thus $A^{[n]} \leq I_n$.

On the other hand, it follows from $AA^T = I_n$ that

$$A^{[n]}(A^T)^{[n]} = \overbrace{AA\cdots A}^{[n]}\overbrace{A^TA^T\cdots A^T}^{[n]} = I_n,$$

that is

$$\sum_{k=1}^{n} a_{ik}^{[n]}a_{ki}^{[n]} = 1.$$

So that

$$a_{ii}^{[n]} = a_{ii}^{[n]} + \sum_{k\neq i} a_{ik}^{[n]} = \sum_{k=1} a_{ik}^{[n]} \geq \sum_{k=1} a_{ik}^{[n]}a_{ki}^{[n]} = 1,$$

i.e., $a_{ii}^{[n]} = 1$. Then $A^{[n]} \leq I_n$ and $a_{ii}^{[n]} = 1$ imply $A^{[n]} = I_n$.

"$\Leftarrow$": If $A^{[n]} = I_n$, then $A^{[n-1]}A = AA^{[n-1]} = I_n$. By the definition of the invertible matrix, matrix A is invertible. This completes the proof.

Proposition 3.2 If $B \in M_n(L)$ is invertible, then $Per(AB) = Per(A)$ for all $A \in M_n(L)$.

Proof If $B \in M_n(L)$ is invertible, then by Theorem 3.1, the rows and columns of B are mutually orthogonal complement. Hence,

$$\begin{aligned}
&\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Big(\sum_{\substack{1\leq i_1,i_2,\cdots,i_n\leq n\\ \exists\, k,l\in L,k\neq l,i_k=i_l}}^{n} a_{1i_1}b_{i_1\sigma(1)}a_{2i_2}b_{i_2\sigma(2)}\cdots a_{ni_n}b_{i_n\sigma(n)}\Big)\\
\leq &\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Big(\sum_{\substack{1\leq i_1,i_2,\cdots,i_n\leq n\\ \forall\, k,l\in L,k\neq l,i_k=i_l}}^{n} b_{i_k\sigma(k)}b_{i_l\sigma(l)}\Big) = 0.
\end{aligned}$$

Thus,

$$\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Big(\sum_{\substack{1\leq i_1,i_2,\cdots,i_n\leq n\\ \exists\, k,l\in L,k\neq l,i_k=i_l}}^{n} a_{1i_1}b_{i_1\sigma(1)}a_{2i_2}b_{i_2\sigma(2)}\cdots a_{ni_n}b_{i_n\sigma(n)}\Big) = 0.$$

In a similar way, we can prove that

$$\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Bigg(\sum_{\substack{1\le i_1,i_2,\cdots,i_n\le n\\ \exists\, k,l\in\underline{n},k\ne l,i_k=i_l}}^{n} a_{1\sigma(1)}b_{\sigma(1)i_1}a_{2\sigma(2)}b_{\sigma(2)i_2}\cdots a_{n\sigma(n)}b_{\sigma(n)i_n}\Bigg)=0.$$

The above two equations together with

$$\sum_{\sigma_2\in\phi(\underline{n},\underline{n})} b_{\sigma_1(i)\sigma_2(i)}=1,$$

for every $i\in\underline{n}$, yield

$$\begin{aligned}
Per(A) &= \sum_{\sigma\in\phi(\underline{n},\underline{n})} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\\
&= \sum_{\sigma_1\in\phi(\underline{n},\underline{n})}\Big\{a_{1\sigma_1(1)}\Big(\sum_{\sigma_2\in\phi(\underline{n},\underline{n})} b_{\sigma_1(1)\sigma_2(1)}\Big)\cdots a_{n\sigma_1(n)}\Big(\sum_{\sigma_2\in\phi(\underline{n},\underline{n})} b_{\sigma_1(n)\sigma_2(n)}\Big)\Big\}\\
&= \sum_{\sigma\in\phi(\underline{n},\underline{n})}\Bigg(\sum_{\substack{1\le i_1,i_2,\cdots,i_n\le n\\ \forall\, k,l\in\underline{n},k\ne l,i_k\ne i_l}}^{n} a_{1i_1}b_{i_1\sigma(1)}a_{2i_2}b_{i_2\sigma(2)}\cdots a_{ni_n}b_{i_n\sigma(n)}\Bigg)\\
&\quad+\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Bigg(\sum_{\substack{1\le i_1,i_2,\cdots,i_n\le n\\ \exists\, k,l\in L,k\ne l,i_k=i_l}}^{n} a_{1i_1}b_{i_1\sigma(1)}a_{2i_2}b_{i_2\sigma(2)}\cdots a_{ni_n}b_{i_n\sigma(n)}\Bigg)\\
&= \sum_{\sigma\in\phi(\underline{n},\underline{n})}\Bigg(\sum_{\substack{1\le i_1,i_2,\cdots,i_n\le n\\ \forall\, k,l\in\underline{n},k\ne l,i_k\ne i_l}}^{n} a_{1i_1}b_{i_1\sigma(1)}a_{2i_2}b_{i_2\sigma(2)}\cdots a_{ni_n}b_{i_n\sigma(n)}\Bigg)\\
&\quad+\sum_{\sigma\in\phi(\underline{n},\underline{n})}\Bigg(\sum_{\substack{1\le i_1,i_2,\cdots,i_n\le n\\ \exists\, k,l\in\underline{n},k\ne l,i_k=i_l}}^{n} a_{1\sigma(1)}b_{\sigma_1(1)i_1}a_{2\sigma(2)}b_{\sigma(2)i_2}\cdots a_{n\sigma(n)}b_{\sigma(n)i_n}\Bigg)\\
&= \sum_{\sigma\in\phi(\underline{n},\underline{n})}\Big(\sum_{k=1}^{n} a_{1k}b_{k\sigma(1)}\Big)\Big(\sum_{k=1}^{n} a_{2k}b_{k\sigma(2)}\Big)\cdots\Big(\sum_{k=1}^{n} a_{nk}b_{k\sigma(n)}\Big)\\
&= Per\Big(\sum_{k=1}^{n} a_{ik}b_{kj}\Big)_{n\times n} = Per(AB).
\end{aligned}$$

This completes the proof.

Remark 3.1 When B is a left multiple to A over a distributive pseudo-lattice, Proposition 3.2 does not hold. For example, consider $(\Omega, \oplus, \otimes)$ in Example 2.1. Let

$$B=\begin{pmatrix} 0 & c\\ c & 0\end{pmatrix},\quad A=\begin{pmatrix} b & 0\\ c & a\end{pmatrix},$$

then

$$BA = \begin{pmatrix} c & a \\ b & 0 \end{pmatrix}. \quad Per(BA) = Per \begin{pmatrix} c & a \\ b & 0 \end{pmatrix} = 0 \neq Per(A) = ba = a.$$

Corollary3.1 If $A \in M_n(L)$ is invertible, then $Per(A) = Per(A^T) = 1$.

Proof If $A \in M_n(L)$ is invertible, then by Proposition 3.1 $A^T A = I_n$. Thus by Proposition 3.2, we have that $Per(A) = Per(A^T A) = Per(I_n) = 1$, i.e., $Per(A) = 1$. And by Proposition 3.1, A^T is also invertible. Hence $Per(A^T) = 1$.

Remark 3.2 Obviously, over a distributive pseudo-lattice L, $Per(A) \neq Per(A^T)$ in general.

Theorem 3.3 Matrix $A \in M_n(L)$ is invertible if and only if $Per(A) = 1$ and the rows and the columns of A are orthogonal.

Proof "$\Rightarrow$": The result follows from Theorem 3.1 and Corollary 3.1.

"$\Leftarrow$": Since

$$1 = Per(A) = \sum_{\sigma \in \phi(\underline{n}, \underline{n})} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)} \leq \sum_{\sigma \in \phi(\underline{n}, \underline{n})} a_{i\sigma(i)} \leq \sum_{k=1}^{n} a_{ik},$$

$$1 = Per(A) = \sum_{\sigma \in \phi(\underline{n}, \underline{n})} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= \sum_{k=1}^{n} \Big\{ \sum_{\substack{\sigma \in \phi(\underline{n}, \underline{n}) \\ \sigma(k) = j}} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)} \Big\}$$

$$\leq \sum_{k=1}^{n} \Big\{ \sum_{\substack{\sigma \in \phi(\underline{n}, \underline{n}) \\ \sigma(k) = j}} a_{kj} \Big\} = \sum_{k=1}^{n} a_{kj}.$$

Hence, we have

$$\sum_{k=1}^{n} a_{ik} = 1, \quad \sum_{k=1}^{n} a_{kj} = 1.$$

Together with the fact that the rows and the columns of A are orthogonal, by Theorem 3.1, A is invertible. This completes the proof.

Proposition 3.3 If $A \in M_n(L)$ is invertible, for every $i, j \in \underline{n}$, there exists orthogonal complement of a_{ij}.

Proof If $A \in M_n(L)$ is invertible, by Theorem 3.1, we have

$$a_{ij} + \sum_{k \neq j} a_{ik} = 1, \quad a_{ij} \sum_{k \neq j} a_{ik} = 0, \quad \Big(\sum_{k \neq j} a_{ik}\Big) a_{ij} = 0 (i, j \in \underline{n}).$$

Hence $\sum\limits_{k \neq j} a_{ik}$ is an orthogonal complement of a_{ij}, similarly, we can prove that $\sum\limits_{k \neq i} a_{kj}$ is also an orthogonal complement of a_{ij}. This completes the proof.

Theorem 3.4 Over an integral distributive pseudo-lattice Z, a matrix A is invertible if and only if it is a permutation matrix.

Proof Let $A \in M_n(L)$, if A is invertible, by Proposition 3.1, $A^{-1} = A^T$. In the following, we will prove that A^T is a permutation matrix. Suppose that A^T is not a permutation matrix. There exists a_{ji} such that $a_{ji} \neq 0$ and $a_{ji} \neq 1$. Let $b = \sum_{k \neq i} a_{jk}$. By Theorem 3.1, we have

$$a_{ji} + b = \sum_{k \in \underline{n}} a_{jk} = 1.$$

So $b \neq 0$. Using Theorem 3.1 again, we have

$$a_{ji}b = a_{ji}\sum_{k \neq i} a_{jk} = \sum_{k \neq i} a_{ji}a_{jk} = 0, \quad ba_{ji} = \Big(\sum_{k \neq i} a_{jk}\Big)a_{ji} = \sum_{k \neq i} a_{jk}a_{ji} = 0,$$

i.e., there exist non zero a_{ji}, $b \in L$ such that a_{ji} and b are mutual orthogonal complement. This contradicts with that Z is an integral distributive pseudo-lattice. This proof is thus completed.

Clearly, The set of all invertible matrices forms a subgroup in $P_n(Z)$.

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分配伪格上的可逆矩阵

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摘　要: 本文介绍了分配伪格上的可逆矩阵，获得分配伪格上矩阵可逆的一些充分必要条件，也证明了在整的分配伪格上，矩阵可逆当且仅当它是一个置换矩阵。这些结果推广了分配格和交换坡上已有关于可逆矩阵的结果。

关键词: 分配伪格；可逆矩阵；正交